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Modulational processes and limits of weak turbulence theory

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Modulational instability of a broad spectrum of Langmuir waves in unmagnetized plasmas is considered. The problem of thresholds of the modulational instability is studied in detail. Applicability limits (which are associated with stochastic properties of turbulent plasma waves) of the theory of weak plasma turbulence are discussed. It it demonstrated that the weak turbulence theory is appropriate for the description of modulational processes developed only for sufficiently wide wave spectra (and/or sufficiently low levels of plasma turbulence). Instability thresholds found on the basis of the theory of weak plasma turbulence are compared with those obtained on the basis of the WKB ansatz, which does not use the random phase approximation. It is demonstrated that for isotropic three-dimensional as well as for one-dimensional Langmuir wave spectra both these approaches result in similar instability conditions and it is shown that for some special spectra of Langmuir waves those are necessary and sufficient (in the mathematical sense) conditions for the development of the modulational instability. Applicability limits of the WKB approximation for the description of the modulational processes are considered in detail and compared with those of the weak turbulence theory.

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I. INTRODUCTION

Modulational instability [1–4] leading to amplitude modulations of a pump monochromatic wave is one of the fundamental effects perturbing wave propagation. Its investigation is important for many practical problems where the coherence of radiation input into plasmas is essential, such as plasma heating in fusion devices [5] or inertial confinement fusion schemes [6,7] and proposed plasma particle accelerators [8], as well as for the interpretation of space and astrophysical data [9,10]. Also, consideration of the modulational processes is necessary for a proper description of the transition from weak to strong turbulence. It is well known that the modulational processes result in the formation of strongly correlated structures (solitonlike, magnetic, etc.) in a plasma [1–4,10,11].

Presently, the theory of weak plasma turbulence [12–14] is well elaborated; the key feature that allows its construction is the so-called random phase approximation applied to Gaussian distributed quantities [13]. Thus this theory is valid only for systems with developed stochasticity [13,14] when wave phases are random, arbitrary wave motion can be represented as a linear superposition of oscillation modes, and the wave packet slowly changes in time due to the interaction with other packets and/or plasma particles. However, such an approach is correct only if the wave amplitudes are sufficiently small. As the amplitudes grow, nonlinear interactions between different modes in the wave packet become significant. In particular, the modulational interaction results in the amplification of phase correlations between different modes of the packet and finally the strongly turbulent state is established with different (compared to weak turbulence) characteristics. The growth of the wave phase correlations can result in the appearance of the coherent structures such as solitons, collapsing wave packets, etc.

It is well known that the character of the modulational instability of a broad wave packet significantly differs from that of a monochromatic pump [15,16] (see also the recent review [17]). In particular, thresholds of the modulational instability of the wave packet can appear. We call here that one monochromatic pump is always modulationally unstable [1–4]. Suppression of the instability of the broad wave packet [16,17] as a result of the modulational interaction of different modes in the packet leads to instability thresholds. In the first approximation, this can be physically understood when the low- (high-) frequency satellite of one mode has opposite phase to the high- (low-) frequency satellite of another mode (of the same packet) and both these satellites have the same frequency. In this case, the satellites can totally disappear (if they have the same amplitudes) and the instability is effectively suppressed.

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Historically it is the investigation of broad wave spectra [18,19] that resulted in the first description of the modulational instability. For example, it has been demonstrated on the basis of the WKB approximation [18] that in the three-dimensional (3D) case, the isotropic spectra of Langmuir oscillations are unstable with respect to density modulations if

$$\int \frac{\omega_{pe}^2 W_k dk}{k^2 v_{Te}^2} > 12 n_0 (T_e + T_i). \tag{1}$$

Thus the sufficient condition for the modulational instability has been established. In (1), n_0 is the nonperturbed plasma density, $T_{e(i)}$ is the electron (ion) temperature, $k = |\mathbf{k}|$, $W_k = 4\pi W_{\mathbf{k}} k^2$, $W_{\mathbf{k}}$ is the spectrum of the Langmuir waves ($W = \int W_{\mathbf{k}} d\mathbf{k}$ is the energy density of plasma oscillations), $\omega_{pe} = (4\pi n_0 e^2/m_e)^{1/2}$ is the electron plasma frequency, e and m_e are the electron charge and mass, and $v_{Te} = (T_e/m_e)^{1/2}$ is the electron thermal speed. Note that the growth rates of the modulational instability were not obtained in [18].

Independently, in Ref. [19] the instability of the "plasmon gas," i.e., packet of random waves, was investigated. However, instability rates obtained there in fact correspond to the case of a narrow wave packet and basically coincide with those for a monochromatic pump.

For the one-dimensional situation (pump waves and modulational perturbations propagate in the same directions), the growth rates have been obtained in [15,20]. These results were also generalized [21,22] on the case of three-dimensional perturbations. In studies [15,20–22], low-frequency plasma motion has been considered.

Recently, a nonlinear formalism allowing us to describe high-frequency plasma perturbations and to investigate the modulational instability for the case of broad wave spectra has been developed [16,17]. One of the most important results of [16] is the absence of instability thresholds in the case of relatively narrow wave spectra. This appears to be reasonable because in the limit of a single monochromatic pump no instability threshold exists. However, the results [16] cannot be applied to the case of broader (in ω space) wave packets, in particular when the instability rate is much less than the characteristic width of the turbulent spectrum (in k space, the latter condition corresponds to a definite connection between the characteristic wave numbers of the modulational perturbations and harmonics in the wave spectrum). At the same time, the case of small rates is the most important when interested in the instability thresholds.

The modulational instability of broad spectra can (to some extent) be modeled by the consideration of two pump waves [17,23–28]. But even in this simplified case the instability is described with the help of a set which consists of an infinite number of coupled equations. To find its solution, one should introduce some simplifying assumptions (for details see [17,23–28]). Naturally, for the case of broad wave spectra the situation is more complicated and (as it has been shown in [16,17]) the modulational instability is described by integral equations which are a generalization of the above set for the instability of two monochromatic pumps.

Furthermore, each mode in the broad wave packet cannot be modulationally unstable independently of other modes. This is the reason why, for the wave packets, we in fact have to study the modulational interaction of different modes in order to describe the modulational instability of the packet as a whole. As a result, for the case of random modes, their correlations are amplified because of the modulational interaction; at the same time, the instability of the broad wave packet is suppressed.

Although it has been realized a sufficiently long time ago [1,3,4] that the modulational interaction leads to the amplification of phase correlations and, consequently, to the transition from the state with the developed wave stochasticity to the strong turbulent state, the exact proof that the modulational interaction necessarily results in such a transition has been presented only recently [29]. In this article, the corresponding range of parameters, where the description of plasma processes based on the theory of weak turbulence is valid, has been established. Moreover, it has been demonstrated that the modulational interaction in the systems with the developed wave stochasticity results in a rapid decrease of the latter.

The understanding of the physical nature of the modulational interactions in broad wave packets has been considerably improved recently. However, the problem of the instability thresholds has not been investigated in detail. Note that because of the fast increase of phase correlations in the modulational processes, this problem is in fact closely connected to the transition from weak turbulence to the strongly turbulent state (and, consequently, with an adequate description of turbulent state of the plasma).

In the present article, the thresholds of the modulational instability of Langmuir waves in collisionless unmagnetized plasma (using recent advances in the theory of the modulational instability and modulational interactions) are found. Applicability limits of the weak turbulence theory are discussed. All calculations (in the framework of the weak turbulence theory) are based on the formalism [16] in which correlation functions of the modes are introduced and integral equations for modulational perturbations of these functions are found. Another approach, which is based on the WKB ansatz (and does not use random phase approximation), is considered in detail. Applicability limits of the WKB approximation are established and compared with those of the weak turbulence theory.

II. EQUATIONS FOR MODULATIONAL INSTABILITY

Here we briefly reproduce the main points of the derivation of equations describing the modulational instability of turbulent spectra [16]. In our consideration, we study random fields in the case of sufficiently weak nonlinearity

$$\frac{W}{n_0 T_e} \ll 1. \tag{2}$$

We separate random (turbulent) and regular components of the electric field $\mathbf{E}$ and the electron distribution function f:

$$\mathbf{E} = \mathbf{E}^{\rm reg} + \delta\mathbf{E}, \quad f = \Phi + \delta f. \tag{3}$$

From the definition of the components we have

$$\langle \delta\mathbf{E} \rangle = 0, \quad \langle \mathbf{E} \rangle = \mathbf{E}^{\rm reg}, \quad \langle \delta f \rangle = 0, \quad \langle f \rangle = \Phi, \tag{4}$$

where the angular brackets $\langle\ \rangle$ denote averaging over a statistical ensemble.

Following the standard procedure [12] of averaging the kinetic equation over a statistical ensemble, we obtain separate equations for the random and regular quantities. Furthermore, we distinguish (see [1,16,17]) the positive- and negative-frequency harmonics E^+ and E^- of the electric field. Taking into account interactions via low-frequency "virtual" (beat) fields E^v and keeping in mind that the condition of the frequency synchronism should be fulfilled, we can thus obtain equations for the low-frequency virtual fields and for the high-frequency wave fields.

Now we have to stress one important point. It is well known that random fields can excite regular virtual fields [15]. Although for homogeneous and isotropic turbulence these regular fields are zero, in the presence of inhomogeneous collective perturbations the nonzero striction force arises, which creates the regular perturbations of the concentration δn and, consequently, leads to the generation of the regular low-frequency electric fields. Thus, as the modulational instability grows, the low-frequency and large-scale regular fields develop [15]; this is the characteristic feature of the turbulent self-organization (note that the nonlinear interaction of the regular fields is also essential in these processes).

We assume that the (nonlinear) spectrum of the weak turbulence $\delta E^{(0)}$ is the solution of the corresponding unperturbed (nonlinear) equation and that the total turbulence field is given by [16,17]

$$\delta E^{\pm} = \delta E^{\pm(0)} + \delta' E^{\pm}, \tag{5}$$

where $\delta' E$ is the modulational perturbation of the turbulent electric field $\delta E^{(0)}$.

In the first approximation the random fields are stationary and homogeneous, i.e., for Fourier components

$$A_k = \frac{1}{(2\pi)^4} \int A(\mathbf{r}, t) \exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}) d\mathbf{r} dt \tag{6}$$

we have [12,16,17]

$$\langle \delta E_k^+ \delta E_{k_1}^- \rangle = -|E^{(0)}|_k^2 \delta(\mathbf{k} + \mathbf{k}_1)\delta(\omega + \omega_1) \tag{7}$$

(note that correlator of two positive-frequency random fields equals zero). The minus sign on the right-hand side (rhs) of Eq. (7) appears because of the definition of the longitudinal field $\mathbf{E}_k = (\mathbf{k}/k)E_k$. Furthermore, the high-frequency regular field does not couple with the low-frequency perturbations. Thus, if the high-frequency regular field is initially equal to zero, then it will be equal to zero at any other moment.

The modulational perturbations are low-frequency ones, so they weakly shift the frequencies of the initial pump wave packet. This means that correlations equal zero for the components of the same frequency sign even in the presence of modulational perturbations. Furthermore, although modulational perturbations are correlated, their correlation to the initial random field cannot be strong because the random character of the pump field is determined independently by initial conditions of the random pumping. Finally, in the linear approximation, only linear perturbations $\delta' E$ are included in the corresponding nonlinear equations.

Now we introduce the following correlation functions:

$$G^{\pm}_{k,k'} = \langle \delta' E^{\pm}_{\pm k + k'} E^{\mp(0)}_{\mp k} \rangle. \tag{8}$$

Equations for these functions are given by [16,17]

$$\begin{aligned}
(\varepsilon_{k+k'} + \varepsilon^N_{k+k'})G^+_{k,k'} = |E^{(0)}|_k^2 \Bigg\{ &\int \frac{[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][(\mathbf{k}_1+\mathbf{k}')\cdot\mathbf{k}_1]}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}+\mathbf{k}'||\mathbf{k}_1+\mathbf{k}'|} \alpha_{k'} G^+_{k_1,k'} d\mathbf{k}_1 d\omega_1 \\
&+ \int \frac{[(\mathbf{k}+\mathbf{k}')\cdot(\mathbf{k}_1+\mathbf{k}')][\mathbf{k}\cdot\mathbf{k}_1]}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}+\mathbf{k}'||\mathbf{k}_1+\mathbf{k}'|} \alpha_{k-k_1} G^+_{k_1,k'} d\mathbf{k}_1 d\omega_1 \\
&+ \int \frac{[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][(\mathbf{k}_1-\mathbf{k}')\cdot\mathbf{k}_1]}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}+\mathbf{k}'||\mathbf{k}_1-\mathbf{k}'|} \alpha_{k'} G^-_{k_1,k'} d\mathbf{k}_1 d\omega_1 \\
&+ \int \frac{[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}_1][(\mathbf{k}_1-\mathbf{k}')\cdot\mathbf{k}]}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}+\mathbf{k}'||\mathbf{k}_1-\mathbf{k}'|} \alpha_{k-k_1+k'} G^-_{k_1,k'} d\mathbf{k}_1 d\omega_1 \Bigg\}
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
(\varepsilon_{-k+k'} + \varepsilon^N_{-k+k'})G^-_{k,k'} = |E^{(0)}|_k^2 \Bigg\{ &\int \frac{[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}][(\mathbf{k}_1-\mathbf{k}')\cdot\mathbf{k}_1]}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}-\mathbf{k}'||\mathbf{k}_1-\mathbf{k}'|} \alpha_{k'} G^-_{k_1,k'} d\mathbf{k}_1 d\omega_1 \\
&+ \int \frac{[(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{k}_1-\mathbf{k}')][\mathbf{k}\cdot\mathbf{k}_1]}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}-\mathbf{k}'||\mathbf{k}_1-\mathbf{k}'|} \alpha_{k-k_1} G^-_{k_1,k'} d\mathbf{k}_1 d\omega_1 \\
&+ \int \frac{[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}][(\mathbf{k}_1+\mathbf{k}')\cdot\mathbf{k}_1]}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}-\mathbf{k}'||\mathbf{k}_1+\mathbf{k}'|} \alpha_{k'} G^+_{k_1,k'} d\mathbf{k}_1 d\omega_1 \\
&+ \int \frac{[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}_1][(\mathbf{k}_1+\mathbf{k}')\cdot\mathbf{k}]}{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}-\mathbf{k}'||\mathbf{k}_1+\mathbf{k}'|} \alpha_{k-k_1-k'} G^+_{k_1,k'} d\mathbf{k}_1 d\omega_1 \Bigg\}.
\end{aligned} \tag{10}$$

In (9) and (10), the factors α are defined by

$$\alpha_k = \begin{cases} \dfrac{1}{4\pi n_0 T_e}\dfrac{k^2 v_s^2}{\omega^2 - k^2 v_s^2} & \text{if } kv_{Ti} \ll \omega \ll kv_{Te}, \\ -\dfrac{1}{4\pi n_0 (T_e + T_i)} & \text{if } \omega \ll kv_{Ti}, \end{cases} \tag{11}$$

where $v_s = (T_e/m_i)^{1/2}$ is the speed of collisionless ion sound and $v_{Ti} = (T_i/m_i)^{1/2}$ is the ion thermal velocity.

Integral equations (9) and (10) are a generalization (for the case of random wave spectra) of the set of two equations for the modulational perturbations of a single monochromatic pump. In the case of two (or many) pump waves the analogous set consists of an infinite number of coupled equations.

To conclude this section, we note that equations for the modulational instability of random fields can be found using another method [12,15,19–22]. In this consideration, the low-frequency perturbations $E^{(1)}$ of the stationary turbulent field $E^{(0)}$ are introduced. All functions $(\Phi, \delta f, E, \delta E)$ are expanded in powers of $E^{(1)}$ and only terms linear on $E^{(1)}$ are considered. Thus two sets of equations for the stationary turbulent state (0) and its perturbation (1) are derived. Then the integrals of the collisions of plasma electrons with turbulent oscillations are calculated. These integrals are present on the rhs of the kinetic equation for the low-frequency perturbation $\Phi^{(1)}$ of (regular) the distribution function. A corresponding technique, which uses basically the same assumptions as above [in particular, correlations functions similar to (8) are introduced], was developed in [12]. Finally, a set of equations analogous to (9) and (10) can be derived.

III. INSTABILITY THRESHOLDS

Using the results of [16], we can significantly simplify the basic set of Eqs. (9) and (10). Indeed, because there is no threshold of the instability in the case of large modulation wave numbers ($k' = |\mathbf{k}'| \sim k_{\text{mod}} \gg k, k_1, k - k_1$), below we neglect k' compared with k, k_1, and $k - k_1$ (i.e., the wave vector of the modulated perturbations is supposed to be smaller than the difference in wave vectors of neighboring modes in the wave packet).

We estimate the relative contribution from terms containing the function α_{k-k_1} with that from terms containing the function $\alpha_{k'}$. From (11) we see that α as a function of k rapidly decreases when $|\omega| \gg kv_s$. At the same time, since we are interested in finding thresholds of the instability, we can assume $|\omega'| \ll k'v_s$, which leads to the maximum possible values of the function α_k. Therefore, if the latter inequality takes place and

$$|\omega - \omega_1| \gg |\mathbf{k} - \mathbf{k}_1| v_s, \tag{12}$$

the terms containing α_{k-k_1} on the rhs of Eqs. (9) and (10) can be neglected. Inequality (12) leads to the condition

$$k_{\text{ch}} \gg k_T = \frac{1}{3r_{\text{De}}}\sqrt{\frac{m_e}{m_i}}, \tag{13}$$

where k_{ch} is the characteristic wave vector of the wave packet and $r_{\text{De}} = v_{Te}/\omega_{pe}$ is the electron Debye length.

The wave number k_T plays a very important role in the weak turbulence theory [12]. In particular, this is the characteristic wave number of differential spectral flow of the Langmuir waves' turbulent energy to the region of small wave vectors. Thus inequality (13) means that for Langmuir waves which are concentrated in the inertial region, the nonlinear frequency shift due to the modulational interactions (i.e., due to terms with $\varepsilon^N_{\mp k+k'}$, α_{k-k_1}, and $\alpha_{k-k_1\pm k'}$) is small (see also [1]). In this case, we obtain, from (9) and (10),

$$\varepsilon_{k+k'} G^+_{k,k'} = |E^{+(0)}|^2_k \alpha_{k'} \int \left(G^+_{k_1,k'} + G^-_{k_1,k'} \right) dk_1 \tag{14}$$

and

$$\varepsilon_{-k+k'} G^-_{k,k'} = |E^{+(0)}|^2_k \alpha_{k'} \int \left(G^-_{k_1,k'} + G^+_{k_1,k'} \right) dk_1. \tag{15}$$

From Eqs. (14) and (15) for the case $|\omega'| \gg k'v_{Ti}$ we find the following dispersion equation for the modulational perturbations (note that we again use $k' \ll k, k_1$):

$$1 = \frac{(k')^2 v_s^2}{(k')^2 v_s^2 - (\omega')^2} \int d\mathbf{k} \frac{\omega_{pe}}{\omega' - \mathbf{k}' \cdot \mathbf{v}_{g,\mathbf{k}}} \left(\mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{k}} \right) \frac{W_{\mathbf{k}}}{4n_0 T_e}, \tag{16}$$

where $\mathbf{v}_g = \mathbf{k} v_{Te}^2/\omega_{pe}$ is the group velocity of the Langmuir wave packet.

For isotropic 3D turbulence and relatively small frequencies such that the inequality

$$k'v_{Ti} \ll |\omega'| \ll k'v_g \tag{17}$$

holds, we obtain

$$(\omega')^2 = (k')^2 v_s^2 \left(1 - \int \frac{W_k dk}{12k^2 r_{\text{De}}^2 n_0 T_e} \right), \tag{18}$$

which gives the instability criterion comparable to (1). However, we stress that Eq. (18) is correct under condition (17) only.

Now we consider the case of smaller rates, namely,

$$|\omega'| \ll k'v_{Ti}. \tag{19}$$

Again, we suppose that 3D turbulence is isotropic. After integration over angles in (16), we find

$$1 = -3(k')^2 v_{Te}^2 \int \frac{dk}{(\omega')^2 - 9k^2 (k')^2 v_{Te}^2 r_{\text{De}}^2} \frac{W_k}{4n_0 (T_e + T_i)}. \tag{20}$$

From this equation, we can clearly see that if

$$|\omega'| > 3k' k_{\max} v_{Te} r_{\text{De}}, \tag{21}$$

where $k_{\max}$ is the maximum wave number of the spectrum W_k, the instability is possible with the rate

$$\gamma_{\text{mod}} = k' v_{Te} \left[\frac{3W}{4n_0(T_e + T_i)} \right]^{1/2} . \tag{22}$$

Here $W = \int dk W_k$ is the energy density of the Langmuir turbulence. Inequality (21) leads to the following condition on the instability development:

$$\frac{W}{n_0(T_e+T_i)} > 12 k_{\max}^2 r_{\text{De}}^2 . \tag{23}$$

At the same time, from (20), we see that if

$$\omega' < 3k' k_{\min} v_{Te} r_{\text{De}}, \tag{24}$$

the instability is absent.

To find a concrete level of wave turbulence that does not satisfy (23), but under which the modulational instability is still possible, we assume (for simplicity) that

$$W_k = \frac{W}{\Delta k} \quad \text{if} \quad k_{\min} < k < k_{\max}, \tag{25}$$

where W =const and $W_k = 0$ for all other wave numbers. Also, we suppose that an inequality opposite (23) takes place, namely,

$$\frac{W}{n_0(T_e+T_i)} < 12 k_{\max}^2 r_{\text{De}}^2 . \tag{26}$$

Then, from Eq. (20) we can conclude that instability is absent if

$$\frac{W}{n_0(T_e+T_i)} < 12 k_{\min}^2 r_{\text{De}}^2 . \tag{27}$$

Furthermore, for the energy level

$$12 k_{\min}^2 r_{\text{De}}^2 < \frac{W}{n_0(T_e+T_i)} < 12 k_{\max}^2 r_{\text{De}}^2, \tag{28}$$

we have, from (20), the dispersion equation

$$(\omega')^2 \left[1 - \frac{1}{12 k_{\max} \Delta k r_{\text{De}}^2} \frac{W}{n_0(T_e+T_i)} \right] = -3(k')^2 v_{Te}^2 \frac{k_{\min}}{4\Delta k} \frac{W}{n_0(T_e+T_i)} . \tag{29}$$

In this case the modulational instability is possible when

$$\frac{W}{n_0(T_e+T_i)} < 12 k_{\max} \Delta k r_{\text{De}}^2, \tag{30}$$

which approximately coincides with (26). Finally, from the assumption

$$3k' k_{\min} v_{Te} r_{\text{De}} < \omega' < 3k' k_{\max} v_{Te} r_{\text{De}}, \tag{31}$$

which has been made to obtain (29), we obtain that the instability holds if

$$12 k_{\min} \Delta k r_{\text{De}}^2 \approx 12 k_{\min} k_{\max} r_{\text{De}}^2 < \frac{W}{n_0(T_e+T_i)} < 12 k_{\max} \Delta k r_{\text{De}}^2 \approx 12 k_{\max}^2 r_{\text{De}}^2 . \tag{32}$$

Thus we can conclude that isotropic Langmuir turbulence with a "flat" spectrum (25) has the following threshold of the modulational instability:

$$\left. \frac{W}{n_0(T_e+T_i)} \right|_{\text{thr}} = 12 k_{\min} k_{\max} r_{\text{De}}^2 . \tag{33}$$

We call here that the only restriction on the possible minimum wave number of the spectrum comes from our assumption $k' \ll k_{\min}$.

Now we consider the one-dimensional case. First of all, we note that under condition (17) we now have instead of (18) the equation

$$(\omega')^2 = (k')^2 v_s^2 \left(1 + \int \frac{W_k dk}{12 k^2 r_{\text{De}}^2 n_0 T_e} \right) . \tag{34}$$

The difference in sign is due to another phase volume of integration in **k** space. From (34), we see that under condition (17) the modulational instability is impossible. When condition (19) holds, we find instead of (20)

$$1 = -3(k')^2 v_{Te}^2 \int \frac{dk}{(\omega' - 3kk' v_{Te} r_{\text{De}})^2} \frac{W_k}{4n_0(T_e+T_i)} . \tag{35}$$

Again, under conditions (21) and (24) we have the same results as in the 3D case.

For spectrum (22) we can integrate (35) to find

$$(\omega - 3k' k_{\min} v_{Te} r_{\text{De}})(\omega - 3k' k_{\max} v_{Te} r_{\text{De}}) = -\frac{3}{4}(k')^2 v_{Te}^2 \frac{W}{n_0(T_e+T_i)} . \tag{36}$$

This equation has the solution

$$\omega = \frac{3}{2} k' v_{Te} \left[(k_{\min} + k_{\max}) r_{\text{De}} \pm \sqrt{(\Delta k)^2 r_{\text{De}}^2 - \frac{W}{3n_0(T_e+T_i)}} \right] . \tag{37}$$

We see that if

$$\frac{W}{n_0(T_e+T_i)} > 3(\Delta k)^2 r_{\text{De}}^2 \approx 3 k_{\max}^2 r_{\text{De}}^2, \tag{38}$$

the instability is possible. For all other levels of turbulence Eq. (37) has no imaginary solutions.

Thus we conclude that the modulational instability of the broad Langmuir wave packet is strongly suppressed in the one-dimensional case (compared with the instability of the isotropic 3D spectrum). We call that for the monochromatic pump wave, its modulational instability is also more efficient in the 3D case, leading to the possibility of Langmuir wave collapse [31]. The same behavior has been observed for narrow wave packets [16,17]. Thus this is common feature of the modulational processes that are mostly effective in 3D situations.

IV. PHASE CORRELATIONS

Here we consider the limits imposed by the assumptions of weak turbulence theory (see also [29]). For simplicity, we investigate wave spectra with $\Delta k \sim k_{\rm ch}$, where Δk is the spectral width.

The main assumption of the weak turbulence theory is that the so-called stochasticity parameter has to be sufficiently large [13]:

$$S \sim \left(\frac{\tau_r}{\tau_K}\right)^{3/4} \gg 1. \tag{39}$$

Here $\tau_r \equiv N\tau_{ac}$ is the recurrence time of correlation of a plasma particle with a wave packet (with the spectral width Δk), $N(\sim \Delta k L/2\pi)$ is the number of modes in the wave packet, L is the characteristic plasma length (which can be considered as the characteristic inhomogeneity scale),

$$\tau_{\rm auto} = \frac{2\pi}{|v_g - v_\phi|\Delta k} \tag{40}$$

is the autocorrelation time within which the correlation of a plasma particle with a wave packet (with the spectral width Δk) is violated, $v_{g(\phi)}$ is the group (phase) velocity of waves in the packet, and

$$\tau_K \sim (k_{\rm ch}^2 D)^{-1/3} \tag{41}$$

is the time of the irreversible decay of the correlations (i.e., of exponential divergence of trajectories and their subsequent mixing in the phase space). The time τ_K is of the order of the inverse Kolmogorov entropy [13,14]. Finally, in (41) D is the quasilinear diffusion coefficient. For the Langmuir waves (under the above assumption $k_{\rm ch} \sim \Delta k$) we have

$$D \sim \frac{W}{n_0 T_e}\omega_{pe} v_{Te}^2. \tag{42}$$

The inequality (39) can be rewritten to the form [29]

$$(k_{\rm ch}L)^{3/4}(k_{\rm ch}r_{\rm De})^{1/2}\left(\frac{W}{n_0T_e}\right)^{1/4} \gg 1, \tag{43}$$

where $r_{\rm De} \equiv v_{Te}/\omega_{pe}$ is the Debye length. Inequality (43) can be fulfilled only if $k_{\rm ch}L \gg 1$ since for Langmuir waves we always have $k_{\rm ch}r_{\rm De} < 1$ as well as $W \ll n_0T_e$, the latter because of (2).

For weakly turbulent processes, the length of the plasma inhomogeneity L in inequality (43) can be estimated from the quasilinear equation as

$$L \sim \frac{v_{Te}^3}{D} \sim r_{\rm De}\frac{n_0T_e}{W}. \tag{44}$$

In this case, we have from (43) the following limit on energy level of Langmuir turbulence:

$$\frac{W}{n_0T_e} \ll (k_{\rm ch}r_{\rm De})^{5/2} \sim (\Delta k r_{\rm De})^{5/2}. \tag{45}$$

If modulational instability develops in the system, then L can be estimated as the characteristic length of plasma density modulations:

$$L \sim L_{\rm mod} \sim \frac{1}{k_{\rm mod}}, \tag{46}$$

where $k_{\rm mod}$ is the characteristic wave vector of the modulational perturbations. From inequality (43) we see that the description of the modulational interactions based on the random phase approximation is valid only for the modulational wave vectors

$$k_{\rm mod} < k_{\rm eff} \sim k_{\rm ch}(k_{\rm ch}r_{\rm De})^{2/3}\left(\frac{W}{n_0T_e}\right)^{1/3}. \tag{47}$$

Note that the rhs of this inequality is always less than $k_{\rm ch}$: $k_{\rm eff} \ll k_{\rm ch}$.

Thus, for the modulational instability developed, the wave numbers of modulated perturbations should be significantly smaller than the spectral width (and/or characteristic wave vector of the packet); see inequality (47). In fact, the latter condition means that the increase of correlations of different modes in the wave packet will be small only in the case when the effective number $N_{\rm eff} \sim k_{\rm eff}L/2\pi \sim k_{\rm eff}/k_{\rm mod}$ of modes (which modulationally interact with each other) does exceed unity. In other words, inequality (47) gives us the condition when instead of the modulational interaction of many modes in the wave packet we have the modulational instability of every separate mode. Only in the case (47) does the system's stochasticity have a level which justifies the use of the random phase approximation and, consequently, the weak turbulence theory.

Condition (43) allows us to determine the limits on the energy density of Langmuir oscillations and their spectral width Δk when the description of the modulational instability on the basis of the theory of weak turbulence is still correct. The characteristic length of the wave vector of modulational perturbations $k_{\rm mod}$ is of order [1–4,30]

$$\frac{W}{n_0T_e} \sim k_{\rm mod}^2 r_{\rm De}^2. \tag{48}$$

Substituting $k_{\rm mod} \sim 1/L$ from Eq. (48) into inequality (43), we find that the description of the modulational interaction on the basis of the theory of weak plasma turbulence is valid only for very wide spectra or for very low turbulent levels in a plasma

$$\frac{W}{n_0T_e} \ll (k_{\rm ch}r_{\rm De})^{10} \sim (\Delta k r_{\rm De})^{10}. \tag{49}$$

In real plasma processes, this inequality can easily be violated.

All of the above considerations have been made for Langmuir turbulent spectra from the inertial region, i.e., when inequality (13) takes place. However, it is well known that other weak turbulence processes that take place for these wave numbers, in particular induced scattering on ions [12], lead to the spectral flow of turbulent energy in the domain of small wave vectors (this is the so-called energy-containing region) when an inequality opposite (13) takes place. In this case we have, from (45) (the spectral width can be estimated as k_T),

$$\frac{W}{n_0 T_e} \ll \left(\frac{m_e}{m_i}\right)^{5/4}. \tag{50}$$

More severe restrictions on the level of turbulence come from Eq. (49)

$$\frac{W}{n_0 T_e} \ll \left(\frac{m_e}{m_i}\right)^{5}. \tag{51}$$

It follows from the latter inequality that for almost every reasonable level of Langmuir turbulence, the weak turbulence theory is inappropriate for the description of the modulational processes developed.

V. WKB ANSATZ

As we have demonstrated above, the random phase approximation imposes severe restrictions on the applicability of the weak turbulence theory. Thus alternative considerations of the modulational processes could be useful.

One possibility is to investigate the so-called multimode modulational instability (as it has been done for two pump waves in [17,23–28]). However, mathematical difficulties do not allow us to find the instability thresholds easily in this case.

A more elegant way has been found in [18] (see also [1]) where a sufficient condition for the modulational instability has been found by considering the change of Langmuir wave energy in the presence of density perturbation. One of the most attractive features of the consideration is that the random phase approximation was not used there. However, there are other limits of applicability, which will be discussed below.

We consider a plasma in a cube of unit volume with a one-dimensional density jump $2\delta n$ in the center (such that the plasma density is $n_0 - \delta n$ if $0 < z < 1/2$ and $n_0 + \delta n$ if $1 > z > 1/2$). The work needed to compress the plasma to this density distribution from an initially homogeneous distribution is given by

$$\delta W_p = \frac{1}{2}\gamma n_0 (T_e + T_i)\left(\frac{\delta n}{n_0}\right)^2, \tag{52}$$

where the factor γ is equal to unity for the isothermal process and 5/3 for the adiabatic process. Expression (52) defines the energy to be given to the plasma. But there is also an energy gain due to the decrease in the energy of the Langmuir waves.

In inhomogeneous plasmas, the frequency of the waves depends on the coordinate

$$\omega_{pe}^2(z) = \omega_{pe}^2 + \delta\omega_{pe}^2(z) = \omega_{pe}^2\left[1 + \frac{\delta n(z)}{n_0}\right]. \tag{53}$$

The dispersion of the Langmuir waves leads to a dependence on z of the z components of their wave vectors

$$k_z(z) = \left(\frac{\omega^2 - \omega_{pe}^2}{3v_{Te}^2} - k_\perp^2\right)^{1/2} = \left(k_z^2 + \frac{\delta\omega^2 - \delta\omega_{pe}^2}{3v_{Te}^2}\right)^{1/2}, \tag{54}$$

where k_z is the value of the function $k_z(z)$ for $\delta n = 0$.

In the WKB approximation the quantity

$$\int \sqrt{k_z^2(z)}\,dz = k_z \tag{55}$$

is conserved. Hence we can consider (55) as an equation that determines $\delta\omega^2$ as a function of δn and k_z. Integration in (55) should be performed only over the transparency region. Some of the waves become trapped at $z < 1/2$. Now we set $k^2 r_{\rm De}^2$ to be much larger than $\delta n/n_0$ and calculate integral (55) in this limit. For nontrapped wave we have

$$\delta\omega_{pe}^2(z) = \omega_{pe}^2\frac{\delta n}{n_0}\begin{cases} -1 & \text{if } z < 1/2 \\ 1 & \text{if } z > 1/2. \end{cases} \tag{56}$$

Furthermore, we use (54) and (55) to find the frequency change of the untrapped waves. The latter is given by

$$\delta\omega_{\rm f}^2 = \frac{1}{12k_z^2 r_{\rm De}^2}\left(\frac{\delta n}{n_0}\right)^2. \tag{57}$$

Note that integration over $z > 1/2$ does not perform if

$$\delta\omega^2 < \omega_{pe}^2\frac{\delta n}{n_0} - 3k_z^2 v_{Te}^2. \tag{58}$$

Putting the frequency shift (57) into (58) we find the critical wave number, which divides the trapped and untrapped waves

$$k_{\rm crit}^2 = \frac{1}{6r_{\rm De}^2}\frac{\delta n}{n_0}. \tag{59}$$

The waves are trapped if $k_z < k_{\rm crit}$ (i.e., they exist only for $z < 1/2$) and untrapped if $k_z > k_{\rm crit}$ (the latter exist for $0 < z < 1$).

Integration of (55) over $z < 1/2$ gives

$$\delta\omega_{\rm tr}^2 = -\omega_{pe}^2\frac{\delta n}{n_0} + 9k_z^2 v_{Te}^2. \tag{60}$$

In the case of the isotropic distribution of Langmuir waves, the waves with the same wave number k but different angles with the z direction have different frequency shifts. The average frequency shift is given by

$$\overline{\delta\omega} = \frac{\overline{\delta\omega^2}}{2\omega_{pe}} = \frac{1}{2\omega_{pe}k}\int_0^k \delta\omega^2(k_z)\,dk_z. \tag{61}$$

Using (61) and (60) we find that the trapped waves always produce a negative frequency shift

$$\overline{\delta\omega_{\rm tr}} = \frac{1}{2\omega_{pe}k}\int_0^k \delta\omega_{\rm tr}^2(k_z)\,dk_z = -\frac{3v_{Te}^2 k_{\rm crit}^3}{2\omega_{pe}k}. \tag{62}$$

The untrapped waves give a frequency shift that is positive, but smaller than (62)

$$\overline{\delta\omega_f} = \frac{1}{2\omega_{pe}k}\int_0^k \delta\omega_f^2(k_z)dk_z = \frac{3v_{Te}^2k_{\text{crit}}^3(k-k_{\text{crit}})}{2\omega_{pe}k^2}. \tag{63}$$

Thus the total frequency shift is negative

$$\overline{\delta\omega} = \overline{\delta\omega_f} + \overline{\delta\omega_{\text{tr}}} = -\frac{3v_{Te}^2k_{\text{crit}}^4}{2\omega_{pe}k^2} = -\frac{\omega_{pe}^3}{24v_{Te}^2k^2}\left(\frac{\delta n}{n_0}\right)^2. \tag{64}$$

Since the number of Langmuir waves is conserved, the change in their energy is due to the change in their frequency

$$\delta W_l = -\left(\frac{\delta n}{n_0}\right)^2\int\frac{W_k dk}{24k^2r_{\text{De}}^2}. \tag{65}$$

The total energy change is given by

$$\delta W = \delta W_p + \delta W_l = \frac{1}{2}\left(\frac{\delta n}{n_0}\right)^2\gamma n_0(T_e+T_i) \times\left[1-\int\frac{W_k dk}{12k^2v_{Te}^2\gamma n_0(T_e+T_i)}\right]. \tag{66}$$

Thus we see that for $\delta W < 0$ the preferred plasma state is inhomogeneous. The factor in square brackets in (66) is remindful of the corresponding factor in (18). However, in contrast to (18), no assumption of the weak turbulence theory has been made to find (66). Also, no conditions on the frequencies and the wave vectors of modulated perturbations such as (17) have been adopted.

It is interesting to note that for spectrum (25), the threshold condition for the development of the modulational instability, which can be found from (66), almost coincides with (33). Indeed, after integration of (66) for spectrum (25), we find that the instability is possible if

$$\frac{W}{\gamma n_0(T_e+T_i)} > 12k_{\min}k_{\max}r_{\text{De}}^2. \tag{67}$$

This coincidence allows us to conclude that at least for spectra such as (25), expression (66) gives us not only the sufficient but also necessary condition for the development of the modulational instability.

The above results cannot be directly applied to the one-dimensional situation since in the latter case we have to use expressions (57) and (60) without averaging over possible values of the angle Θ between the propagation of high-frequency Langmuir waves and low-frequency density modulations. Incidentally, the assumption of isotropicity of the wave spectrum has not been used in deriving frequency shifts $\delta\omega_f^2$ and $\delta\omega_{\text{tr}}^2$. Thus we can use some of the results obtained above. We have

$$\delta W_l = \frac{1}{2\omega_{pe}^2}\int(\delta\omega_f^2+\delta\omega_{\text{tr}}^2)W_{\mathbf{k}}d\mathbf{k}. \tag{68}$$

Integrating over dk_xdk_y (taking into account that $\delta\omega_f^2 + \delta\omega_{\text{tr}}^2$ depends on only k_z) and introducing

$$W_{k_z} = 2\int W_{\mathbf{k}}dk_xdk_y, \tag{69}$$

we convert to the one-dimensional problem. Note that on the rhs of (69), a factor 2 appeared because we have taken into account negative values of k_z (thus we assume that the function W_{k_z} is even in k_z; in all subsequent formulas integration is performed only over positive k_z).

Furthermore, if we substitute in (68) the concrete expressions for $\delta\omega_f^2$ and $\delta\omega_{\text{tr}}^2$, introduce the characteristic scale k_* of change of the one-dimensional spectrum W_{k_z} (so that we can consider $W_{k_z} \approx$ const for $k < k_*$), and choose $k_{\text{crit}} \ll k_*$, we find

$$\delta W_l = \frac{1}{2}\left(-\frac{1}{k_*}W_{k_z}|_{k_z\sim k_*} + \int_{k_*}^{\infty}dk_z\frac{W_{k_z}}{k_z^2}\right)\frac{1}{12r_{\text{De}}^2}\left(\frac{\delta n}{n_0}\right)^2. \tag{70}$$

The second term (which contains the integral) in (70) is positive. The first term on the rhs in (70) is negative due to the contribution of trapped waves.

If we have a flat spectrum (25), we find from (70) and (52) that the modulational instability is possible under the condition

$$\frac{W}{\gamma n_0(T_e+T_i)} > 12k_{\max}^2r_{\text{De}}^2, \tag{71}$$

which is similar to (38). We call that result (70) contains a negative contribution into δW_l only if there are trapped waves (in other words, if the turbulent spectrum contains waves with $k_z < k_{\text{crit}}$).

If all of the wave numbers in the spectrum W_{k_z} are larger than k_{crit} (i.e., if $k_{\min} > k_{\text{crit}}$), then we have from (70) [compare with (66)]

$$\delta W = \frac{1}{2}\left(\frac{\delta n}{n_0}\right)^2\gamma n_0(T_e+T_i) \times\left[1+\int\frac{W_{k_z}dk_z}{12k_z^2v_{Te}^2\gamma n_0(T_e+T_i)}\right]. \tag{72}$$

This expression has a factor (in square brackets) that is very similar to that in (34). However, there is no modulational instability in this case.

VI. APPLICABILITY LIMITS OF THE WKB APPROXIMATION

In the above derivation some steps need clarification. In particular, when we integrate (in 3D case) over all possible harmonics to calculate δW_l, we have to take into account different characteristics of waves depending on whether their k_z is more or less than k_{crit}. Indeed, expression (63) takes place if $k > k_{\text{crit}}$; if $k < k_{\text{crit}}$, we do not have untrapped waves at all. This means that on the rhs of (63) the step function $\theta(k-k_c)$ appears, which will effect subsequent integration. Moreover, Eq. (62) takes place also only if $k > k_{\rm crit}$; otherwise we have integration until k [not $k_{\rm crit}$ as in (62)] and in the numerator on the rhs of this equation the expression $2k_{\rm crit}^2 k - k^3$ will appear instead of $k_{\rm crit}^3$. Thus the above consideration is correct for isotropic spectra only when

$$k_{\rm min} > k_{\rm crit}. \tag{73}$$

If the opposite inequality holds, we have to distinguish intervals of $k < k_{\rm crit}$ and $k > k_{\rm crit}$ when integrating over k in the corresponding expression for δW_l. We stress that condition (73) does not mean that we have no untrapped waves, since in the considered geometry it is necessary to have $k_z = k\cos\Theta < k_{\rm crit}$ for the wave to be trapped. The latter inequality can easily be fulfilled even for waves whose wave numbers satisfy (73).

Moreover, the above use of the WKB approximation can be inadequate when considering the step function of $\delta n(z)$. In fact, we have to smoothen this step, e.g., as $\delta n(z) = n_0 \tanh(Ak_z z)$, where $A \ll 1$. Furthermore, we can integrate over z to some L_0 which should be much larger than the characteristic scale of change of the smoothed function. That is, we have to adopt $L_0 \gg 1/Ak_z$. In this case, the assumption $A \ll 1$ implies that the WKB approximation can work (with a small inhomogeneity on the scale of the wavelength) and, on the other hand, the inequality $L_0 \gg 1/Ak_z$ means that we can treat the considered function as almost a step function (in other words, the corrections to the result of the step function will be at least of order $1/Ak_zL_0$).

However, the point is that in the development of the modulational instability we probably cannot assume that δn is an arbitrary, sufficiently smooth function. At the same time, for narrow spectra we have to obtain (in the limit $\Delta k \to 0$) the results for one monochromatic pump, i.e., in the absence of threshold. In this sense, the WKB threshold obviously fails.

If we assume that the width of the transition region is L, then for justification of WKB approach we should require the following: (a) L does not make a significant contribution to the solution of the corresponding equation for $\delta\omega^2$ and (b) the values of k_z for which the WKB approximation is not applicable (i.e., $k_zL \le 1$) do not make a significant contribution to the integral determining $\overline{\delta\omega}$.

Requirement (a) results in an inequality, which is defined by (here we do not use our previous assumption of the unit length of the system and introduce $L_0 \neq 1$)

$$\int_{L_0/2-L/2}^{L_0/2+L/2} \sqrt{k_z^2 + \frac{\delta\omega^2}{3v_{Te}^2}}\, dz \ll k_z. \tag{74}$$

From (74) we then obtain

$$\left(\frac{\delta n}{n_0}\right)^2 \frac{L}{L_0} \ll 36 k_z^4 r_{\rm De}^4. \tag{75}$$

Requirement (b) leads to the following: the values of k_z for which the WKB approximation is not applicable (i.e., $k_zL \le 1$) are not significant if $1/L \ll k_{\rm crit}$, i.e., when

$$L^2 \gg 6 r_{\rm De}^2 \frac{n_0}{\delta n}. \tag{76}$$

Conditions (75) and (76) can be fulfilled simultaneously only if

$$\frac{\delta n}{n_0} \ll 6 k_z^2 r_{\rm De}^2 (k_z L_0)^{2/3}. \tag{77}$$

Since we are interested in the process of the near-threshold development of the modulational instability, we can require the initial $\delta n/n_0$ to be sufficiently small in order to satisfy inequality (77) [however, (76) should be maintained]. Note that inequality (77) has been derived without any assumptions like $k \sim k_{\rm mod}$.

If the characteristic scale of the plasma inhomogeneity is determined by processes of weak turbulence (e.g., quasilinear diffusion), then from (44) and (76) we find

$$\frac{W}{n_0T_e} \ll \left(\frac{\delta n}{n_0}\right)^{1/2} \sim k_{\rm crit} r_{\rm De}. \tag{78}$$

Because of (73) we then obtain that the WKB approximation is correct in this case if

$$\frac{W}{n_0T_e} \ll k_{\rm ch} r_{\rm De}, \tag{79}$$

which is obviously not as strong as (45).

We obtain a further estimation using inequality (77) together with (48) and

$$\frac{W}{n_0T_e} \sim \frac{\delta n}{n_0} \tag{80}$$

(the latter is a consequence of the dynamic equation describing slow plasma motion [31]). This means that the WKB approach is applicable for

$$k_{\rm mod} \ll k_{\rm ch}(k_{\rm ch}L_0)^{1/3}(\cos\Theta)^{4/3}. \tag{81}$$

Thus the considered wave spectrum cannot be concentrated in the regions $k \to 0$ and/or $\Theta \to \pi/2$ because in this case (81) can be violated. The limit $\Theta = \pi/2$ corresponds to the "perpendicular" development of the modulational instability, when $\mathbf{k}_{\rm mod} \perp \mathbf{k}$. Consideration of the instability of two monochromatic pumps demonstrates the absence of thresholds in this case [17,23–28].

For the modulational instability developed, we substitute L from (46) and use (76) to find

$$\frac{W}{n_0T_e} \gg \frac{\delta n}{n_0}. \tag{82}$$

However, this inequality contradicts (80). Thus we see that the WKB ansatz is inappropriate for any turbulence level when developed modulational instability exists (and the corresponding inhomogeneity scale is determined by the modulational processes). We can conclude that in this case, the instability threshold in fact determines the applicability limits of the theory.

VII. CONCLUSION

The most important results of the paper are expressions (18) and (33) for 3D isotropic spectra and (38) for

1D Langmuir turbulent spectra obtained on the basis of the theory of weak plasma turbulence, as well as (66) and (67) for 3D spectra and (71) for 1D spectra found using the WKB approximation. A detailed analysis of the applicability limits of the approaches used allows us to conclude that both the above descriptions can be used to study the near-threshold behavior of the modulational instability of broad turbulent spectra. However, for the modulational instability developed when the characteristic inhomogeneity scale is determined by the modulational processes, the theory of weak plasma turbulence has severe applicability limits [see, in particular, (49)] and the WKB approximation cannot be used.

Thus we can conclude that within their applicability limits both the weak turbulence theory and the WKB approximation give basically the same expressions for the thresholds of the modulational instability of broad wave packets. At the same time, we have found that in the case when the characteristic inhomogeneity scale is determined by the quasilinear diffusion, more severe conditions on plasma parameters are imposed by assumptions of stochastic wave properties than those of the WKB approximation; therefore the WKB approximation could be valid for the description of the near-threshold behavior of the modulational instability for more types of broad wave spectra.

The results obtained have been found under the assumption that the wave number of the modulational perturbations are much less than the difference in wave numbers of any two harmonics in the turbulent spectrum: $k' \sim k_{\rm mod} \ll k - k_1$. However, the following question arises: Can the thresholds found be the real thresholds of the modulational instability? In other words, can the instability develop if, e.g., $k_{\rm mod} \gg k - k_1$? The answer is given in [16]: In the latter situation the instability develops and there are no thresholds in the 3D isotropic case as well as the 1D case. However, in this case there are limitations on the possible wave numbers of the modulational perturbations, in particular

$$k^2_{\rm mod} \leq \frac{1}{r^2_{\rm De}} \frac{W}{n_0 T_e}. \tag{83}$$

This inequality, together with $k_{\rm mod} \gg \Delta k$, can be realized when

$$\frac{W}{n_0 T_e} \gg \Delta k^2 r^2_{\rm De}. \tag{84}$$

Comparing (84) with, e.g., (33) or (38), we see that the latter expressions are indeed thresholds since they indicate instability for a pump level that is not larger than (84). Of course, we still have no answer in the case when the wave number of the modulated perturbations is of the order of the difference between the wave numbers of harmonics in the turbulent spectrum. This problem is the most difficult for investigation. Here we note only that for the two monochromatic pumps the analogous problem of the modulational instability when the wave number of the modulated perturbations is of the order of the difference between wave numbers of the pumps is also not solved yet; see, e.g., [17].

To summarize, we have considered the modulational instability of the broad spectrum of Langmuir waves in unmagnetized plasmas. To consider the packets of random waves, we have used the formalism [16] developed for the description of the modulational instability on the basis of the theory of weak plasma turbulence. We have determined the limits of applicability of this theory for the description of the modulational processes. We have shown that the theory of weak plasma turbulence is valid only for the description of the modulational processes of sufficiently wide wave spectra and/or sufficiently low levels of plasma turbulence.

We have also considered the problem of the modulational instability thresholds on the basis of the approach [18] that uses the WKB approximation and does not apply the random phase approximation. We have demonstrated that both these approaches (based on the weak turbulence theory and on the WKB ansatz) give similar threshold conditions for the case of isotropic three-dimensional wave spectra. Moreover, for the case of "flat" wave spectra this threshold condition is coincident with the necessary and sufficient (in the mathematical sense) condition for the development of the modulational instability.

More complicated is the one-dimensional situation (when all wave vectors have the same direction). We have demonstrated that the description of the modulational effects based on the theory of weak plasma turbulence allows us to conclude that the modulational instability of the one-dimensional broad Langmuir wave packet is strongly suppressed compared with the instability of the isotropic three-dimensional spectrum (and consequently the instability has higher thresholds than that in the 3D isotropic case). Consideration of the modulational instability of the one-dimensional wave spectra on the basis of the WKB approximation gives similar results (in particular, approximately the same thresholds).

Here we have investigated the simplest case of collisionless unmagnetized plasmas. The case considered is important for proper construction of basic principles of the theory of the modulational interaction of broad wave spectra. We also note that the detailed studies of the modulational processes for broad wave spectra (and, in particular, investigation of applicabilty limits of different theories) for more complicated situations (e.g, in magnetized, collision-dominated plasmas, etc.) are also of significant interest. This would be useful for the interpretation of different phenomena in space and astrophysical plasmas (collisionless shocks, pulsar emission, solar bursts, solar flares, solar wind, cosmic rays, etc.), plasmas of Earth's ionosphere and planetary atmospheres, as well as in laboratory plasmas (plasmas of nuclear fusion devices, laser plasmas, etc.).

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[1] L.I. Rudakov and V.N. Tsytovich, Phys. Rep. C **40**, 1 (1978).
[2] S.G. Thornhill and D. ter Haar, Phys. Rep. C **43**, 43 (1978).
[3] M.V. Goldman, Rev. Mod. Phys. **56**, 709 (1984).
[4] V.D. Shapiro and V.I. Shevchenko, in *Basic Plasma Physics*, edited by A.A. Galeev and R.N. Sudan (North-Holland, Amsterdam, 1984), Vol. II.
[5] V.E. Golant and V.I. Fedorov, *High-Frequency Methods of Plasma Heating in Toroidal Nuclear Fusion Devices* (Energoatomizdat, Moscow, 1986).
[6] W.L. Kruer, *Physics of Laser Plasma Interactions* (Addison-Wesley, Redwood City, CA, 1988).
[7] E.M. Campbell, Phys. Fluids B **4**, 3781 (1992).
[8] T. Tajima and J.M. Dawson, Phys. Rev. Lett. **43**, 267 (1979).
[9] D.B. Melrose, *Instabilities in Space and Laboratory Plasmas* (Cambridge University Press, Cambridge, 1986).
[10] D. ter Haar and V.N. Tsytovich, Phys. Rep. C **73**, 175 (1981).
[11] S.V. Vladimirov and S.I. Popel, Phys. Scr. **50**, 161 (1994).
[12] V.N. Tsytovich, *Theory of Turbulent Plasma* (Consultants Bureau, New York, 1977).
[13] J. Krommes, in *Basic Plasma Physics* (Ref. [4]).
[14] R.Z. Sagdeev, D.A. Usikov, and G.M. Zaslavsky, *Nonlinear Physics. From the Pendulum to Turbulence and Chaos* (Harwood, Chur, 1988).
[15] V.N. Tsytovich, Zh. Eksp. Teor. Fiz. **57**, 141 (1969) [Sov. Phys. JETP **30**, 83 (1970)].
[16] S.I. Popel, V.N. Tsytovich, and S.V. Vladimirov, Phys. Plasmas **1**, 2176 (1994).
[17] S.V. Vladimirov and S.I. Popel, Aust. J. Phys. **47**, 375 (1994).
[18] A.K. Gailitis, Ph.D. thesis, P.N. Lebedev Institute, 1964; Izv. Latv. SSR Ser. Phys. Tech. Nauk **4**, 13 (1965).
[19] A.A. Vedenov and L.I. Rudakov, Dokl. Akad. Nauk SSSR **159**, 767 (1964) [Sov. Phys. Dokl. **9**, 1073 (1965)].
[20] K. Komilov, F.Kh. Khakimov, and V.N. Tsytovich, Fiz. Plazmy **5**, 35 (1979) [Sov. J. Plasma Phys. **5**, 20 (1979)].
[21] K. Komilov, F.Kh. Khakimov, and V.N. Tsytovich, Zh. Tekh. Fiz. **46**, 1388 (1976) [Sov. Phys. Tech. Phys. **21**, 790 (1976)].
[22] K. Komilov, F.Kh. Khakimov, and V.N. Tsytovich, Fiz. Plazmy **4**, 1302 (1978) [Sov. J. Plasma Phys. **4**, 727 (1978)].
[23] S.V. Vladimirov and V.N. Tsytovich, Zh. Eksp. Teor. Fiz. **98**, 1279 (1990) [Sov. Phys. JETP **71**, 715 (1990).
[24] V.N. Tsytovich and S.V. Vladimirov, Comments Plasma Phys. Controlled Fusion **15**, 39 (1992).
[25] S.V. Vladimirov and V.N. Tsytovich, Phys. Lett. A **171**, 360 (1992).
[26] S.V. Vladimirov and V.N. Tsytovich, J. Plasma Phys. **49**, 197 (1993).
[27] S.V. Vladimirov and V.N. Tsytovich, J. Plasma Phys. **49**, 207 (1993).
[28] S.V. Vladimirov and V.N. Tsytovich, Fiz. Plazmy **19**, 1132 (1993) [Plasma Phys. Rep. **19**, 592 (1993)].
[29] S.I. Popel and S.V. Vladimirov (unpublished).
[30] Generally speaking, this is not necessarily the case. Indeed, in the simplest situation when the instability is described by the nonlinear Schrödinger equation, this condition implies that the nonlinear term is of the order of the linear dispersion term. But the situation is possible when (at least at some stage of the modulational instability development) the nonlinear term can be of the order of the time derivative term and dispersion effects can be negligible.
[31] V.E. Zakharov, in *Basic Plasma Physics* (Ref. [4]).